**Graph Theory and Applications**
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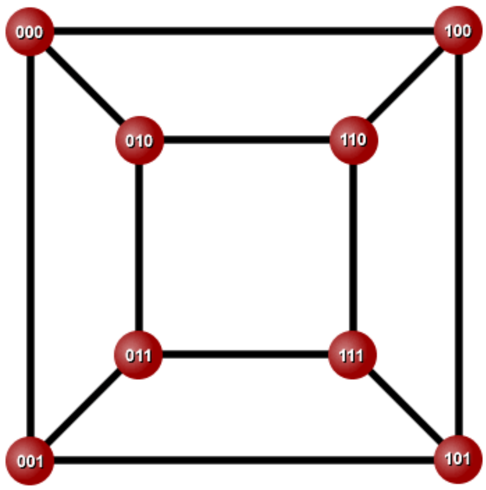
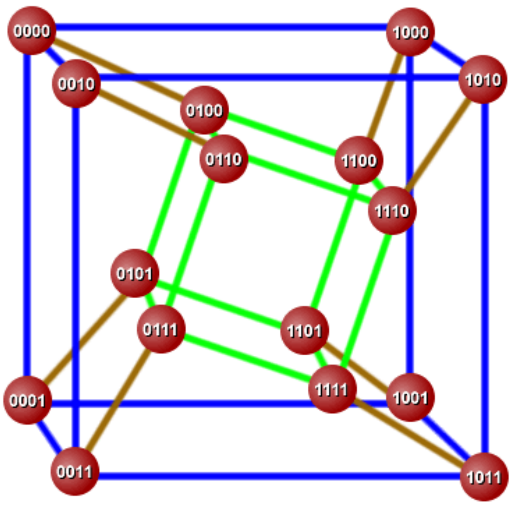
**Homework 1** **10.10.2008**

**1.** The k-cube is the graph whose vertices are the ordered k-tuples of 0's and 1's, two vertices being joined if and only if they differ in exactly one coordinate.
**a.** Draw the 3-cube and 4-cube.
**b.** Show that k-cube has $k.2^{k-1}$ edges, and is bipartite.

**a.**

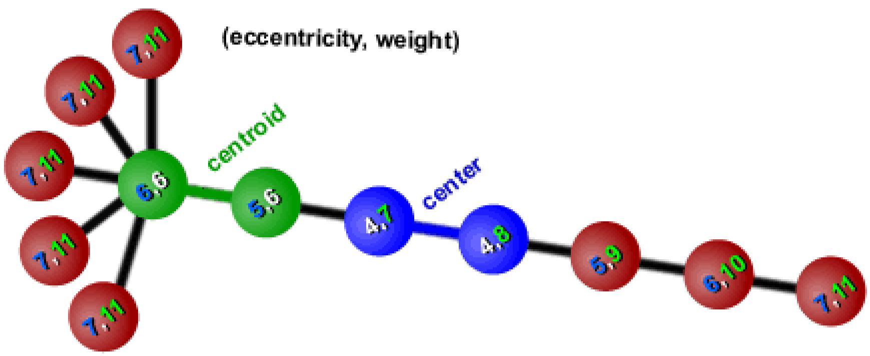



**b.** The first proof is direct. In k-cube, each vertex has exactly k edges, therefore; each vertex could be joined to all of its neighbor vertices (that differs in exactly one coordinate). Using the fact that; there are $2^k$ vertices in k dimensional coordinate system, we have $k.2^k$ edges as a sum. However, we have counted all edges twice for both of its neighbor vertices. As a result, k-cube has totally $k.2^k / 2 = k.2^{k-1}$ edges.

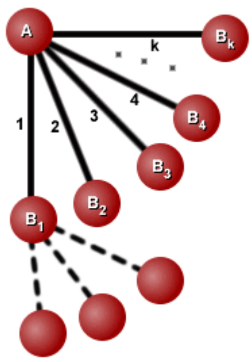
Assume that we have divided the vertices of G into subsets A and B which have the following properties. Set A has the vertices that have odd number of 1's in their k-tuples and set B has the others (that have even number of 1's). It is always possible to divide a k-cube into two disjoint subsets like that. Since, none of the subset has dual vertices that differ in exactly one coordinate, all the edges connect one vertex from A to another vertex from B. As a result, we could always bipartite the k-tube graph.

**2.** Draw a tree with disjoint center and centroid, both containing two vertices.

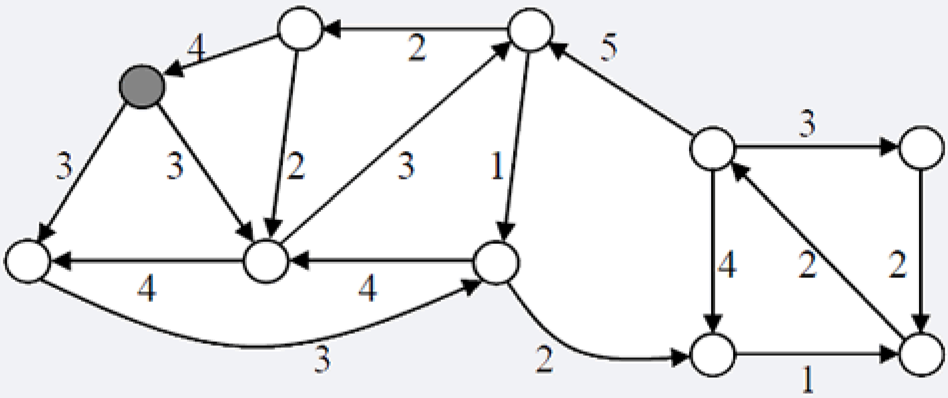


**3.** The girth of G is the length of a shortest cycle in G; if G has no cycles we define the girth of G to be infinite. Show that a k-regular graph of girth four has at least 2k vertices.

Assume that the k-regular graph G has a girth of four and $B_i$ ($i = 1, 2, \ldots, k$) are the adjacent vertices of A. Under these circumstances, dual $B_s$ and $B_t$ ($s, t \in i$) could not be adjacent, because of girth requirement. Therefore, $B_s$ should have k-1 more adjacent vertices $C_j$ ($j = 1, 2, \ldots, k-1$). As a result, we have at least $A_1 + B_k + C_{k-1} = 2k$ vertices.

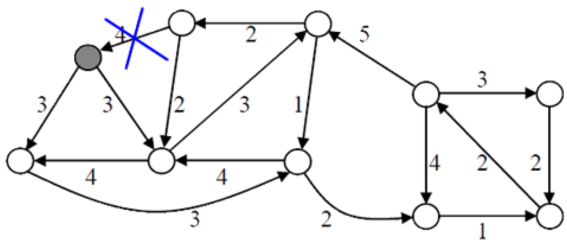



**4.** Find a directed minimum spanning tree for the graph on the right. Choose gray colored vertex as the root.

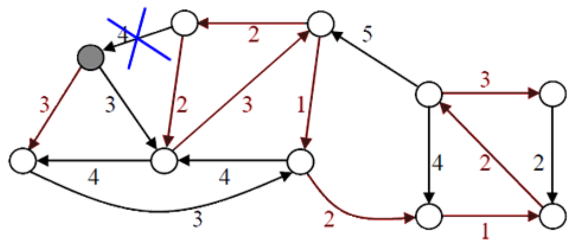


We will use Chu-Liu / Edmonds algorithm to solve the given directed minimum spanning tree problem.

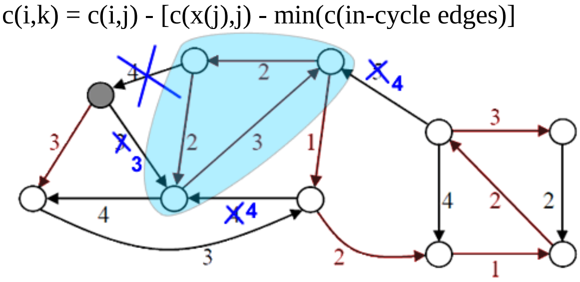
a. Discard the arcs entering the root if any.



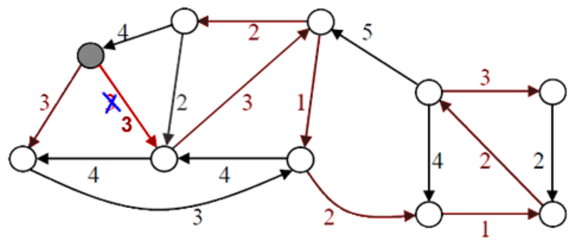
b. Select the entering arc with the smallest cost.



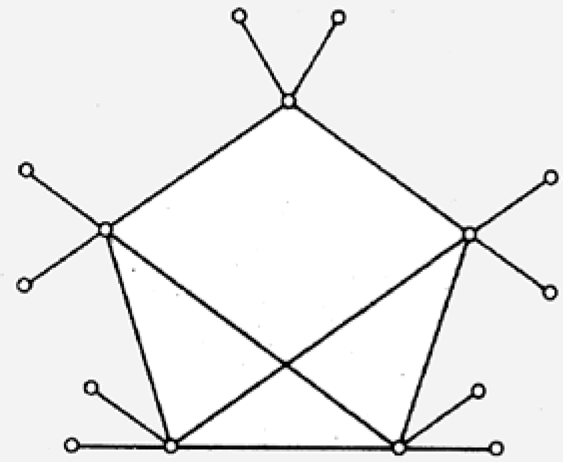
c. For each cycle formed: Contract the nodes in the cycle into a pseudo-node k & modify the cost of arcs as
$c(i,k) = c(i,j) - [c(x(j),j) - \min(c(\text{in-cycle edges})]$

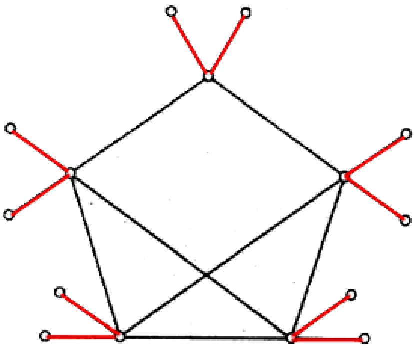


d. Select the entering arc with the smallest modified cost. Replace the arc in S (to same real node) by the new selected arc.



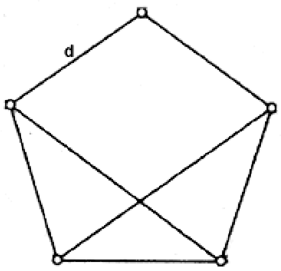
**5.** Find the number of non-isomorphic spanning trees in the graph on the right.



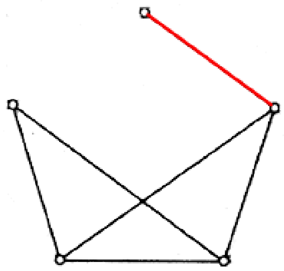


There is no need to analyze red edges emphasized in G because they do have to be used in all non-isomorphic spanning trees. Therefore, studying with the graph on the left is isomorphic to working with the $G_0$ graph given below.

We will use the recursive method that works with contraction as shown below to solve the problem.

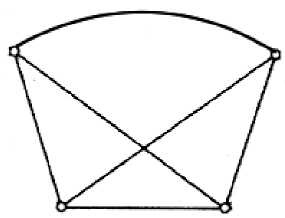



$G_0$

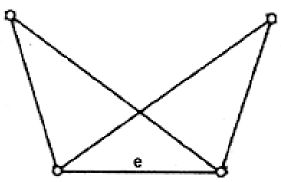


$G_0 - d : G_1$

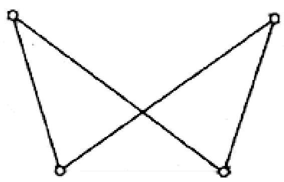
+



$G_0 \cdot d : n^{n-2} = 4^2 = 16$




$G_2$



$G_2 - e$ : 4 spanning trees

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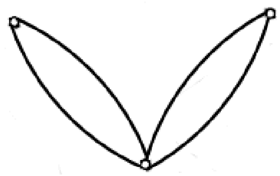

$G_2 \cdot e$ : 4 spanning trees

$G_2$ has 8 non-isomorphic spanning trees. It is obvious that $G_1 \approx G_2$.
Therefore, $G_1$ has 8 non-isomorphic spanning trees too.
Then, $G_0$ has 8 + 16 = 24 non-isomorphic spanning trees. Again, it is clear that $G \approx G_0$.
As a result, there are 24 non-isomorphic spanning trees in G.